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Effect of electrical polarization saturation on stress intensity factors in a piezoelectric ceramic

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Abstract

The electric-field induced stress intensity factor in a piezoelectric medium of limited electrical polarization is evaluated based on a strip-saturation model of the Dugdale-type. Particular emphasis is placed on the effect of the saturation condition on the near tip field and the stress intensity factor. To this end, the general solution is derived in terms of the (unspecified) normal electrical displacement distribution along the saturated strip. Since the saturated strip is representative of the unknown saturated zone, the normal electrical displacement may suffer discontinuity across the saturated strip. It is found that the crack-tip field and the stress intensity factors depend on the discontinuity of the normal electrical displacement across the saturated strip although this dependency disappears in some practically important cases. A crack perpendicular to the poling axis in a general poled ferroelectric is discussed in detail to illustrate the implications of the strip-saturation model for electric-field induced cracking. The results show that some discrepancies between theory and experiments, for which the classical linear piezoelectric model gives qualitatively incorrect results, can be explained clearly in terms of the stress intensity factor given by the strip-saturation model. In particular, these results are independent of the form of the saturation condition imposed on the saturated strip.

1. Introduction

Recently, motivated by some discrepancies between theory and experiments on fracture behavior of piezoelectrics/ferroelectrics, several important issues have been raised in the literature (e.g., see Pisarenko et al., 1985; Mehta and Virkar, 1990; Pak, 1990, 1992; Suo et al., 1992; Tobin and Pak, 1993; Pak and Tobin, 1993; Cao and Evans, 1994; Park and Sun, 1995). Much of these issues is focused on the effects of an electrical field on the crack growth in a piezoelectric material. Analysis based on linear piezoelectric model predicts that an electrical field does not induce any non-zero stress intensity factor (Pak, 1990, 1992; Suo, 1993). In particular, this implies that an insulating crack should never grow under an electric-field load. However, the electric-field induced crack growth has been observed substantially along the direction perpendicular to the applied electrical field (Cao and Evans, 1994; Lynch et al., 1995). Further, under combined electrical and mechanical

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loading, linear piezoelectric model predicts that an electrical field, irrespective of its sign, impedes the growth of a crack in a poled ferroelectric. However, experiments (see Tobin and Pak, 1993; Park and Sun, 1995) have shown that the crack growth along the direction perpendicular to the poling axis is enhanced by application of an electrical field in the same direction as the poling, and retarded by an electrical field opposite to the poling direction.

In an effort to remove these theoretical difficulties, it is widely recognized that the dielectric nonlinearity near the crack tip plays an essential role. Based on the fact that (see Sundar and Newnham, 1992) the electrical polarization is achieved by interior ionic movement and the latter is restricted by a limited state, an idealized model of polarization saturation has been applied to isotropic dielectrics by a number of authors, see, e.g. Yang and Suo (1994), Lynch et al. (1995), Hao et al. (1996), Gong and Suo (1996), Ru et al. (1997), Ru and Mao (1997), where the nonlinear aspect of the polarization saturation has been handled in a manner analogous to plastic yielding. More recently, by employing a strip-saturation model of the Dugdale-type (Dugdale, 1960), an effort has been made by Gao et al. (1997) (see also Gao and Barnett, 1996) to explore the role of the polarization saturation in a poled ferroelectric. Their results show that the polarization saturation offers a plausible explanation for the aforementioned discrepancies. However, Gao et al. (1997) have based their discussions on a simplified model (for instance, they have ignored one of two degrees of freedom of the displacement field and the corresponding boundary conditions, and have considered a very special set of material constants), and therefore the validity of their conclusions for a general poled ferroelectric remains unproved.

The present work is motivated by an attempt to explore the role of the polarization saturation in a general piezoelectric medium. Of particular interest is the effect of the saturation condition on the stress intensity factors. The solution procedure, described in Section 3, leads to explicit expressions for the stress intensity factors. In Section 4, the implications of the strip-saturation model for the abovementioned discrepancies are discussed in terms of the electric-field induced stress intensity factor. Some interesting predictions of the strip-saturation model are pointed out which need further experimental verification.

2. Formulation

The basic equations for a piezoelectric are given by

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0$$

$$\gamma_{ij} = \tfrac{1}{2}[u_{i,j}+u_{j,i}], \quad E_i = -\phi_{,i}$$

$$\sigma_{ij} = C_{ijkl}\gamma_{kl} - e_{kij}E_k, \quad D_k = e_{kij}\gamma_{ij} + \varepsilon_{kl}E_l \tag{1}$$

where u_i and ϕ are the displacements and electrical potential, σ_{ij}, γ_{ij}, E_i and D_i are the stresses, strains, electrical field and electrical displacements, and C_{ijkl}, e_{ijk} and ε_{ij} are the elastic, piezoelectric and dielectric constants, respectively. In the two-dimensional case (see Stroh, 1958; Suo et al., 1992), we consider the solution of the form

$$u(x,y) \equiv \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \\ u_3(x,y) \\ \phi(x,y) \end{bmatrix} = af(x+py) \tag{2}$$

where $f(*)$ is an analytic function, p a complex number and a a constant four-element column. The equations (1) are satisfied by arbitrary $f(*)$ if

$$(c_{ijkl}a_k + e_{lij}a_4)f_{,jl} = 0, \quad (e_{jkl}a_k - \varepsilon_{jl}a_4)f_{,jl} = 0, \quad i,k = 1,2,3, \quad j,l = 1,2 \tag{3}$$

For existence of a non-zero vector a, p has to satisfy an eigen-equation. For a stable material, the eight eigen-roots form four conjugate pairs with non-zero imaginary parts. Assume that p_α are four distinct roots with positive imaginary parts and a_α ($\alpha = 1,2,3,4$) the associated eigen-vectors, the general solution of (1) can be given in the form

$$u(x,y) = \sum_1^4 [a_\alpha f_\alpha(z_\alpha) + \overline{a_\alpha f_\alpha(a_\alpha)}], \quad z_\alpha = x + p_\alpha y, \quad \alpha = 1,2,3,4 \tag{4}$$

and the associated stresses and electrical displacements are given by

$$(\sigma_{2i}, D_2) = \sum_1^4 [b_\alpha f'_\alpha(z_\alpha) + \overline{b_\alpha}\,\overline{f'_\alpha(z_\alpha)}], \quad (\sigma_{1i}, D_1) = -\sum_1^4 [b_\alpha p_\alpha f'_\alpha(z_\alpha) + \overline{b_\alpha p_\alpha}\,\overline{f'_\alpha(z_\alpha)}] \tag{5}$$

where $f_\alpha(*)$ are four arbitrary analytic functions, and each of four column vectors b_α ($\alpha = 1,2,3,4$) is determined by the corresponding pair (p, a) through

$$b_j = (C_{2jkl}a_k + e_{lj2}a_4)(x+py)_{,l} = 0, \quad b_4 = (e_{2kl}a_k - \varepsilon_{2l}a_4)(x+py)_{,l} = 0 \tag{6}$$

Thus, the problem is reduced to determining the four analytic functions $f_\alpha(*)$ ($\alpha = 1,2,3,4$) such that all mechanical and electrical boundary conditions are satisfied. In the paper, for convenience, we adopt the notations (see Suo et al., 1992)

$$A = (a_1, a_2, a_2, a_4), \quad B = (b_1, b_2, b_3, b_4), \quad f(z) = (f_1(z), f_2(z), f_3(z), f_4(z))^T$$
$$B_p = (p_1 b_1, p_2 b_2, p_3 b_3, p_4 b_4), \quad Y \equiv iAB^{-1}, \quad H = Y + \bar{Y} \tag{7}$$

3. The strip-saturation model

As mentioned previously, the basic idea adopted in this paper is that dielectric nonlinearity is essential to a physically more realistic description of the crack tip behavior in piezoelectric materials. Further, if the perfect saturation model is employed, the magnitude of the electrical displacement should be bounded from above by a given material constant. Thus, we have

$|D| \leqslant$ a given constant

and the electrical displacement is "saturated" when the equality holds. In the paper, no further discussion will be given for the construction of constitutive theory consistent with the above saturation condition. Instead, the strip-saturation model of the Dugdale type will be employed to investigate the effect of limited electrical polarization on the stress intensity factors in piezoelectric materials.

Since linear piezoelectric model gives singular electrical displacement at the crack tip, it is

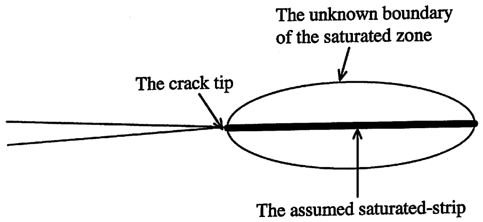

Fig. 1. The strip-model of the Dugdale type.

expected that there will be a saturated zone near the crack tip, as shown in Fig. 1. Exact determination of the saturated zone in plane deformation is extremely difficult. Hence, following Dugdale (1960), we assume that the unknown saturated zone around the crack tip can be approximately replaced by a segment directly ahead of the crack tip, as shown in Fig. 1. Hence, when a finite crack situated on the interval $L = [-a, a]$ is considered, it is assumed that the electrical displacement is saturated on two segments $[-c, -a]$ and $[a, c]$, denoted by S, as shown in Fig. 2. Now, a relevant saturation condition must be imposed on the saturated strip S. Of course, one can choose the saturation condition: $|D| =$ constant along the saturated strip. However, since the saturated strip is representative of the unknown saturated zone, this saturation condition is not the only reasonable choice. In fact, the saturation condition on the strip S is somewhat uncertain (obviously, this uncertainty reflects the approximate character of the strip-saturation model). In recognition of this fact, the results which are independent of the form of the saturation condition are of particular interest. To express a general saturation condition, we consider the saturation condition given in terms of arbitrary normal electrical displacement, $D_2^+(x)$ and $D_2^-(x)$, along S. Hence, the boundary value problem of the strip saturation model for a finite crack is of the form

$$\sigma_{2i} = 0, \quad D_2 = 0, \quad |x| < a, \quad y = 0$$

$$u_i^{(+)} = u_i^{(-)}, \quad \sigma_{2i}^{(+)} = \sigma_{2i}^{(-)}, \quad D_2^+ = D_2^+(x), \quad D_2^- = D_2^-(x), \quad a < |x| < c, \quad y = 0$$

$$\sigma_{2i} \to \sigma_{2i}^{\infty}, \quad \sigma_{1i} \to \sigma_{1i}^{\infty}, \quad D_2 \to D_2^{\infty}, \quad D_1 \to D_1^{\infty}, \quad |z| \to \infty, \quad i = 1, 2, 3 \tag{8}$$

where the subscripts "+" and "−" indicate the values from the upper and lower half-planes, and σ_{ij}^{∞} and D_i^{∞} ($i = 1, 2, j = 1, 2, 3$) are the remote loading parameters.

Of course, for chosen saturation condition (for instance, $|D| =$ constant), the corresponding

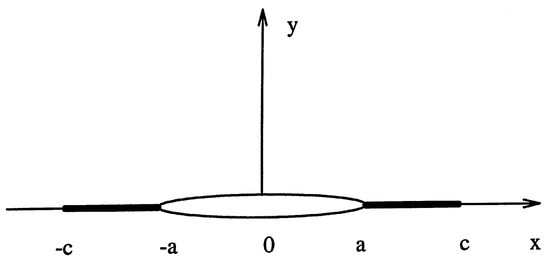

Fig. 2. The strip-saturation model for a finite crack in a piezoelectric medium.

values of $D_2^+(x)$ and $D_2^-(x)$ are unknown in general. However, because the electrical displacement is saturated on the strip S, its magnitude (and then $D_2^+(x)$ and $D_2^-(x)$) must be bounded along S including the crack tips. The merit of the present method is that, as will be seen below, the crack tip field and the stress intensity factors can be exactly determined, even without any further detail of $D_2^+(x)$ and $D_2^-(x)$, provided the (unknown) normal electrical displacement is bounded on the strip S.

3.1. *The general solution*

First, we derive the explicit form of the general solution. From (5), (8), the continuities of traction along the crack faces L and the saturated strip S and of the normal electrical displacement along L give

$$Bf'(x^+)+\bar{B}\overline{f'}(x^-) = Bf'(x^-)+\bar{B}\overline{f'}(x^+)+(0,0,0,\Delta D_2(x))^T, \quad |x|<c \tag{9}$$

where $\Delta D_2 = (D_2^+ - D_2^-)$ denotes the discontinuity of D_2 across the saturated strip S (which vanishes on L). The condition (9) can be written in an equivalent form

$$[Bf'(x)-\bar{B}\overline{f'}(x)]^+ - [Bf'(x)-\bar{B}\overline{f'}(x)]^- = (0,0,0,\Delta D_2(x))^T, \quad |x|<c \tag{10}$$

Note that all functions appearing on the left of (10) are analytic in the whole plane except $L+S$ and approach constants at infinity, it follows that

$$Bf'(z)-\bar{B}\overline{f'}(z) = iT_0 + C(z), \quad C(z) \equiv \frac{1}{2\pi i}\int_s \frac{(0,0,0,\Delta D_2(t))^T}{t-z}\,\mathrm{d}t \tag{11}$$

in the entire plane, where T_0 is a constant real column and the Cauchy integral $C(z)$ is analytic in the entire plane except S. In view of (5), (8) and (11), we find

$$Bf'(z) \to \tfrac{1}{2}[(\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty)^T + iT_0] \tag{12}$$

On using (11), the remaining boundary conditions on the crack faces L give

$$[Bf'(x)]^+ + [Bf'(x)]^- = iT_0 + C(z), \quad |x|<a, \quad y=0 \tag{13}$$

Further, let

$$\phi(x^+)-\phi(x^-) = g(x), \quad a<|x|<c, \quad y=0 \tag{14}$$

the displacement continuity condition along the saturated strip S give

$$[Af'(x)-\bar{A}\overline{f'}(x)]^+ - [Af'(x)-\bar{A}\overline{f'}(x)]^- = (0,0,0,g'(x))^T, \quad a<|x|<c, \quad y=0 \tag{15}$$

In view of (7), (11), (15) can be written as

$$H[Bf'(x)]_-^+ = \bar{Y}(0,0,0,\Delta D_2(x))^T + (0,0,0,ig'(x))^T, \quad a<|x|<c, \quad y=0 \tag{16}$$

Finally, the saturation condition on S gives

$$[Bf'(x)]^+ + [Bf'(x)]^- = (*,*,*,D_2^+(x))^T + iT_0 + C(z)^-, \quad a<|x|<c, \quad y=0 \tag{17}$$

where "*" denotes some arbitrary quantities.

Obviously, the four analytic functions $f_\alpha(*)$ ($\alpha = 1, 2, 3, 4$) can be determined if the single-variable analytic function $Bf(z)$ is known. Define

$$Bf'(z) \equiv \frac{1}{\sqrt{z^2-a^2}} h(z) + i\frac{T_0}{2} + \frac{1}{2}C(z), \quad h(z) = \begin{Bmatrix} h_1(z) \\ h_2(z) \\ h_3(z) \\ h_4(z) \end{Bmatrix} \tag{18}$$

It is seen from (13) that $h(z)$ is continuous across the crack faces L and analytic in the whole plane except the saturated strip S. From (12), (16)–(18), $h(z)$ is now determined by the following conditions

$$\begin{aligned} &[h(x)]^+ + [h(x)]^- = \sqrt{(x^2-a^2)}(*, *, *, \tfrac{1}{2}(D_2^+(x) + D_2^-(x)))^T, \quad a < |x| < c, \quad y = 0; \\ &[h(x)]^+_- = \sqrt{(x^2-a^2)}[H^{-1}(0,0,0,ig'(x))^T + [H^{-1}(\bar{Y}-Y)](0,0,0,\tfrac{1}{2}\Delta D_2(x))^T], \\ &a < |x| < c, \quad y = 0; \quad h(z) \to \tfrac{1}{2}(\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty)^T z + (0,0,0,\mathrm{const})^T, \quad |z| \to \infty \end{aligned} \tag{19}$$

where the first three components of the constant column appearing on the last line of (19) must be zero for (18) to be the derivative of a single-valued analytic function in the far field limit. It is seen from (19) that the analytic function $h_4(z)$ can be determined in a decoupled way by

$$\begin{aligned} &[h_4(x)]^+ + [h_4(x)]^- = \tfrac{1}{2}(D_2^+(x) + D_2^-(x))\sqrt{(x^2-a^2)}, \quad a < |x| < c, \quad y = 0, \\ &h_4(z) \to \tfrac{1}{2}D_2^\infty z + \mathrm{const}, \quad |z| \to \infty \end{aligned} \tag{20}$$

Once $h_4(z)$ is known, the electric potential jump $g(x)$ can be found from (19) as

$$\begin{aligned} &h_4(x^+) - h_4(x^-) = \sqrt{(x^2-a^2)}[ig'(x)[H^{-1}]_{44} + \tfrac{1}{2}[H^{-1}(\bar{Y}-Y)]_{44}\Delta D_2(x)], \\ &a < |x| < c, \quad y = 0 \end{aligned} \tag{21}$$

Thus, the other three functions are determined by

$$\begin{aligned} &[h_i(x)]^+_- = \frac{1}{2}\sqrt{x^2-a^2}(D_2^+ - D_2^-)\left[[H^{-1}(\bar{Y}-Y)]_{i4} - \frac{[H^{-1}]_{i4}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\right] \\ &+ \frac{[H^{-1}]_{i4}}{[H^{-1}]_{44}}[h_4(x)]^+_-, \quad a < |x| < c, \quad y = 0, \quad i = 1, 2, 3 \\ &h_i(z) \to \frac{1}{2}\sigma_{2i}^\infty z, \quad |z| \to \infty \end{aligned} \tag{22}$$

with the result

$$h_i(z) = \frac{[H^{-1}]_{i4}}{[H^{-1}]_{44}} h_4(z) + q(z)\left[[H^{-1}(\bar{Y}-Y)]_{i4} - \frac{[H^{-1}]_{i4}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\right]$$

$$+\frac{1}{2}\left[\sigma_{2i}^{\infty}-D_2^{\infty}\frac{[H^{-1}]_{i4}}{[H^{-1}]_{44}}\right]z, \quad i=1,2,3 \tag{23}$$

where $q(z)$ is a Cauchy integral defined by

$$q(z)=\frac{1}{4\pi i}\int_S \frac{(D_2^+(t)-D_2^-(t))\sqrt{t^2-a^2}}{t-z}\,\mathrm{d}t \tag{24}$$

Hence, $Bf'(z)$ is given as follows

$$Bf'(z)=i\frac{T_0}{2}+\frac{1}{2}C(z)+\frac{1}{\sqrt{z^2-a^2}}\left\{h_4(z)\begin{Bmatrix}[H^{-1}]_{14}/[H^{-1}]_{44}\\ [H^{-1}]_{24}/[H^{-1}]_{44}\\ [H^{-1}]_{34}/[H^{-1}]_{44}\\ 1\end{Bmatrix}+\frac{z}{2}\begin{Bmatrix}\sigma_{21}^{\infty}-D_2^{\infty}[H^{-1}]_{14}/[H^{-1}]_{44}\\ \sigma_{22}^{\infty}-D_2^{\infty}[H^{-1}]_{24}/[H^{-1}]_{44}\\ \sigma_{23}^{\infty}-D_2^{\infty}[H^{-1}]_{34}/[H^{-1}]_{44}\\ 0\end{Bmatrix}\right.$$

$$\left.+q(z)\begin{Bmatrix}[H^{-1}(\bar{Y}-Y)]_{14}-\frac{[H^{-1}]_{14}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ [H^{-1}(\bar{Y}-Y)]_{24}-\frac{[H^{-1}]_{24}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ [H^{-1}(\bar{Y}-Y)]_{34}-\frac{[H^{-1}]_{34}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ 0\end{Bmatrix}\right\} \tag{25}$$

where the real column T_0 should be determined by the asymptotic values of other stress and electrical displacement components (this will not be discussed in the paper).

We now determine $h_4(z)$. Because $Bf'(z)$ exhibits only an inverse square-root singularity at the two crack tips and cannot exhibit any singularity at the points $x=c$ and $x=-c$, $h_4(z)$ cannot exhibit any singularity at all. Thus, using the standard method (Muskhelishvili, 1963), $h_4(z)$ is given by

$$h_4(z)=\frac{\sqrt{(z+c)(z+a)(z-a)(z-c)}}{2\pi i}\int_S \frac{\frac{1}{2}(D_2^+(t)+D_2^-(t))}{(t-z)(\sqrt{(t+c)(t-c)})^+}\,\mathrm{d}t \tag{26}$$

with the condition (for the existence of the solution)

$$\frac{D_2^\infty}{2}+\frac{1}{2\pi i}\int_S \frac{\left[\frac{1}{2}(D_2^+(t)+D_2^-(t))\sqrt{(t^2-a^2)}\right]}{(\sqrt{(t+c)(t+a)(t-a)(t-c)})^+}\,\mathrm{d}t=0 \tag{27}$$

which determines the size of the saturated segments. Thus, the global solution of (8) is given by (25) in terms of the normal electrical displacement on the saturated strip S.

For example, if one assumes that (see Gao et al., 1997; Gao and Barnett, 1996)

$$D_2^+(x)=D_2^-(x)=D_s,\quad a<|x|<c \tag{28}$$

it follows that $C(z)=q(z)=0$ and then $Bf'(z)$ has an elementary form as follows

$$Bf'(z)=\frac{z}{2\sqrt{z^2-a^2}}\begin{Bmatrix}\sigma_{21}^\infty-D_2^\infty[H^{-1}]_{14}/[H^{-1}]_{44}\\ \sigma_{22}^\infty-D_2^\infty[H^{-1}]_{24}/[H^{-1}]_{44}\\ \sigma_{23}^\infty-D_2^\infty[H^{-1}]_{34}/[H^{-1}]_{44}\\ 0\end{Bmatrix}+i\frac{T_0}{2}$$

$$+\frac{D_s}{2\pi}\left[\pi+\arccos\left(\frac{c^2+az}{c(a+z)}\right)-\arccos\left(\frac{c^2-az}{c(z-a)}\right)\right]\begin{Bmatrix}[H^{-1}]_{14}/[H^{-1}]_{44}\\ [H^{-1}]_{24}/[H^{-1}]_{44}\\ [H^{-1}]_{34}/[H^{-1}]_{44}\\ 1\end{Bmatrix} \tag{29}$$

An equivalent form of (29) has been given by Gao and Barnett (1996) for the strip model (28). The solution (29) gives the stress intensity factors as follows

$$\begin{pmatrix}K_{\mathrm{II}}\\ K_{\mathrm{I}}\\ K_{\mathrm{III}}\end{pmatrix}=\sqrt{\pi a}\begin{pmatrix}\sigma_{21}^\infty-D_2^\infty[H^{-1}]_{14}/[H^{-1}]_{44}\\ \sigma_{22}^\infty-D_2^\infty[H^{-1}]_{24}/[H^{-1}]_{44}\\ \sigma_{23}^\infty-D_2^\infty[H^{-1}]_{34}/[H^{-1}]_{44}\end{pmatrix} \tag{30}$$

Despite its reasonable aspect, an assumption associated with the model (28) is that the normal electrical displacement is continuous across the saturated strip S. Note that because the saturated strip is representative of the unknown saturated zone, electrical quantities may suffer discontinuities across the saturated strip. Although the fact that there is no charge within the saturated zone implies

$$\int_{\pm a}^{\pm c}[D_2^+(x)-D_2^-(x)]\,\mathrm{d}s=0$$

there is no reason to assert that the jump of $D_2(x)$ across the strip S should be ignored in general. This suggests that the discontinuity of the normal electrical displacement, which is ignored in (28), may play a significant role in the strip-saturation model. In any case, because the saturated strip is just an idealization of the unknown saturated zone, it is necessary to consider other physically more relevant saturation conditions (for instance, $|D|$ = constant) which allow for discontinuity of the normal electrical displacement across the strip. It is this idea that motivates the present study.

3.2. *The stress intensity factors*

We turn to the general model (8). Let us consider the crack tip field at $x = a$. Note that $h_4(a) = 0$ and $q(a)$ is finite (because the normal electrical displacement is bounded, see (24), (26)); (25) gives the dominant-order crack tip field at the crack tip $x = a$

$$Bf'(z) = \frac{1}{\sqrt{2a(z-a)}}\left\{\frac{a}{2}\begin{Bmatrix}\sigma_{21}^{\infty}-D_2^{\infty}\dfrac{[H^{-1}]_{14}}{[H^{-1}]_{44}}\\ \sigma_{22}^{\infty}-D_2^{\infty}\dfrac{[H^{-1}]_{24}}{[H^{-1}]_{44}}\\ \sigma_{23}^{\infty}-D_2^{\infty}\dfrac{[H^{-1}]_{34}}{[H^{-1}]_{44}}\\ 0\end{Bmatrix}\right.$$
$$\left. + q(a)\begin{Bmatrix}[H^{-1}(\bar{Y}-Y)]_{14}-\dfrac{[H^{-1}]_{14}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ [H^{-1}(\bar{Y}-Y)]_{24}-\dfrac{[H^{-1}]_{24}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ [H^{-1}(\bar{Y}-Y)]_{34}-\dfrac{[H^{-1}]_{34}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ 0\end{Bmatrix}\right\} \quad (31)$$

In particular, the stress intensity factors are given by

$$\begin{pmatrix}K_{\mathrm{II}}\\ K_{\mathrm{I}}\\ K_{\mathrm{III}}\end{pmatrix} = \sqrt{\pi a}\begin{Bmatrix}\sigma_{21}^{\infty}-D_2^{\infty}\dfrac{[H^{-1}]_{14}}{[H^{-1}]_{44}}\\ \sigma_{22}^{\infty}-D_2^{\infty}\dfrac{[H^{-1}]_{24}}{[H^{-1}]_{44}}\\ \sigma_{23}^{\infty}-D_2^{\infty}\dfrac{[H^{-1}]_{34}}{[H^{-1}]_{44}}\end{Bmatrix}$$
$$+2\sqrt{\frac{\pi}{a}}q(a)\begin{Bmatrix}[H^{-1}(\bar{Y}-Y)]_{14}-\dfrac{[H^{-1}]_{14}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ [H^{-1}(\bar{Y}-Y)]_{24}-\dfrac{[H^{-1}]_{24}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\\ [H^{-1}(\bar{Y}-Y)]_{34}-\dfrac{[H^{-1}]_{34}}{[H^{-1}]_{44}}[H^{-1}(\bar{Y}-Y)]_{44}\end{Bmatrix} \quad (32)$$

It is noted that there is only an unknown imaginary number $q(a)$ in (31) and (32), which represents the effect of the discontinuity of normal electrical displacement on the crack tip field. In particular, it is seen from (31), (32) that the sum of the upper and the lower normal electrical displacements on the saturated strip, $D_2^+(x)+D_2^-(x)$, has no effect on the crack tip field. This result implies the "invariance" shown by Gao and Barnett (1996) for the model (28).

Hence, without solving the full problem (8), the crack-tip field and the stress intensity factors can be determined to within an imaginary coefficient $q(a)$, which is related to the discontinuity of the normal electrical displacement through (24). Because the electrical displacement field is saturated on S, for any physically relevant saturation condition, the associated tangential electrical displacement should not exhibit a square-root singularity as the crack tip is approached along S. From (5), (7), (31), this condition requires that the real part of the fourth component of the column

$$(B_p B^{-1})(Bf'(z)), \quad z = x > a \tag{33}$$

vanishes, where $Bf'(z)$ is given by (31). In general, because the four roots p_α ($\alpha = 1, 2, 3, 4$) are distinct, the condition (33) will not be satisfied automatically. If that is the case, (33) provides the condition to determine the value of $q(a)$ and then the exact expressions for the crack tip field and the stress intensity factors can be obtained without specifying the form of the saturation condition. This interesting issue will not be discussed further in the paper because, as will be seen below, the mode-I stress intensity factor is found to be independent of $q(a)$ for the crack perpendicular to the poling axis in a poled ferroelectric, which is the focus of the present study.

4. Discussion

In this section, we examine the implications of the strip-saturation model (8) for electric-field induced cracking in a general poled ferroelectric. Although the energy release rate should be regarded as a more reasonable criterion for fracture of electroelastic solids, the stress intensity factor has widely been used for convenience as the fracture criteria for piezoelectric/ferroelectric materials (see e.g. Yang and Suo, 1994; Lynch et al., 1995; Hao et al., 1996; Gong and Suo, 1997). We now show that the discrepancies between theory and experiments mentioned before can be explained clearly by the strip-saturation model in terms of the electric-field induced stress intensity factor.

We use the constitutive relations for a poled ferroelectric given in, e.g., Pak (1992), Suo et al. (1992) or Park and Sun (1995), where the poling direction is chosen as the positive direction of the axis-3. Consider plane-strain deformation of a finite crack in the [1–3] plane; then,

$$u_2 \equiv 0, \quad \gamma_{12} = \gamma_{22} = \gamma_{23} = 0, \quad E_2 = D_2 = 0 \tag{34}$$

and the constitutive relations reduce to the forms

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & c_{44} \end{pmatrix} \begin{pmatrix} \gamma_{11} \\ \gamma_{33} \\ 2\gamma_{13} \end{pmatrix} - \begin{pmatrix} 0 & e_{31} \\ 0 & e_{33} \\ e_{15} & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_3 \end{pmatrix}$$

$$\begin{pmatrix} D_1 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & e_{15} \\ e_{31} & e_{33} & 0 \end{pmatrix} \begin{pmatrix} \gamma_{11} \\ \gamma_{33} \\ 2\gamma_{13} \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} & 0 \\ 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_3 \end{pmatrix}$$

$$(c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, \varepsilon_{11}, e_{33}, \varepsilon_{33}, e_{15}) > 0, \quad c_{11}c_{33} > c_{13}^2, \quad e_{31} < 0 \tag{35}$$

and

$$\sigma_{21} = \sigma_{23} = 0, \quad \sigma_{22}(x, y) = c_{12}\gamma_{11} + c_{13}\gamma_{33} - e_{31}E_3 \tag{36}$$

As mentioned before, almost all of the current discrepancies focus on the fracture behavior of a crack perpendicular to the poling axis in a poled ferroelectric. In this case, the real symmetric matrix H is of the form (see Suo et al., 1992)

$$H = 2\begin{pmatrix} 1/C_L & 0 & 0 \\ 0 & 1/C_T & 1/e \\ 0 & 1/e & -1/\varepsilon \end{pmatrix}, \quad C_T > 0, \quad e > 0, \quad \varepsilon > 0 \tag{37}$$

where C_T, e and ε represent the elastic moduli, piezoelectricity and permittivity, respectively. It follows from (37) that

$$[H^{-1}]_{14}/[H^{-1}]_{44} = 0, \quad [H^{-1}]_{34}/[H^{-1}]_{44} = \frac{-C_T}{e} < 0 \tag{38}$$

Further, the following anti-symmetric matrix can be written as

$$Y - \bar{Y} = 2i\begin{pmatrix} 0, & Im[Y_{13}], & Im[Y_{14}] \\ -Im[Y_{13}], & 0, & Im[Y_{34}] \\ -Im[Y_{14}], & -Im[Y_{34}], & 0 \end{pmatrix} \tag{39}$$

Hence, from (37), (39), we have

$$[H^{-1}(\bar{Y} - Y)]_{14} = -iC_L Im[Y_{14}], \quad [H^{-1}(\bar{Y} - Y)]_{34} = -i\frac{e^2 C_T}{e^2 + \varepsilon C_T} Im[Y_{34}],$$

$$[H^{-1}(\bar{Y} - Y)]_{44} = -i\frac{\varepsilon e C_T}{e^2 + \varepsilon C_T} Im[Y_{34}] \tag{40}$$

Obviously, these matrices represent the properties of a piezoelectric material and depend not on the loading condition. Hence, it is possible to determine some of their elements by considering certain special loading conditions. For instance, let us consider the mode-I mechanical load (with $D_3^\infty = 0$). In this case, note that

$$c_{44}D_1 = (\varepsilon_{11}c_{44} + e_{15}^2)E_1 + e_{15}\sigma_{13} \tag{41}$$

the coefficient of the dominant-order tangential electrical displacement directly ahead of the crack tip is proportional to

$$(\varepsilon_{11}c_{44}+e_{15}^2)Im[Y_{34}]\sigma_{33}^{\infty} \tag{42}$$

According to the numerical solutions for this case, see e.g. Pak (1992), Sosa (1992) and Kumar and Singh (1996), the dominant-order tangential electrical displacement vanishes directly ahead of the crack tip; it follows from (42) that

$$Im[Y_{34}] = 0 \tag{43}$$

Hence, from (32), (38), (40), (43), the strip-saturation model gives the mode-I stress intensity factor at the tip of a crack perpendicular to the poling axis as

$$K_{\mathrm{I}} = \sqrt{\pi a}\left[\sigma_{33}^{\infty} + \frac{C_T}{e}D_3^{\infty}\right] \tag{44}$$

which is independent of $q(a)$. It should be noted that the model (28) gives the same result as (44) in this special case. What is proven by the present work is that the result (44) is independent of the form of the saturation condition. This fact provides a much stronger basis for the result (44).

The first term on the right of (44) is identical to the mode-I stress intensity factor predicted by linear piezoelectric model (without polarization saturation), and therefore the second term represents the effect of the polarization saturation on the mode-I stress intensity factor. Because the fracture of mode-I is of primary interest, we now examine the implications of (44) for the electric-field induced crack growth in a general poled ferroelectric. To our knowledge, this issue has not been examined for a general poled ferroelectric, especially in terms of the stress intensity factor.

We now consider several cases separately ($\sigma_{13}^{\infty} = 0$ for all cases).

The electrical field is parallel to the crack axis

In this case, (35) and (44) give

$$K_{\mathrm{I}} = \sqrt{\pi a}\left(\sigma_{33}^{\infty} + \frac{C_T(c_{11}e_{33}-c_{13}e_{31})}{e(c_{11}c_{33}-c_{13}^2)}\sigma_{33}^{\infty} + \frac{C_T}{e}\frac{(c_{33}e_{31}-c_{13}e_{33})}{(c_{11}c_{33}-c_{13}^2)}\sigma_{11}^{\infty}\right) \tag{45}$$

Hence, the electrical field parallel to the crack faces has no influence on the mode-I stress intensity factor. This is in agreement with the experimental fact that an electrical field parallel to the crack faces has no significant effect on the crack growth. Further, because the coefficient of the second term on the right of (45) is positive, the strip-saturation model predicts a mode-I stress intensity factor bigger than that predicted by a linear piezoelectric model (in the absence of the stress σ_{11}^{∞}). This is consistent with the experimental result that the crack perpendicular to the poling axis has a lower fracture toughness (see Pisarenko et al., 1985; Mehta and Virkar, 1990).

In the presence of the stress, σ_{11}^{∞}, parallel to the crack axis, (45) predicts that σ_{11}^{∞} has a negative (or positive) contribution to the mode-I stress intensity factor if its sign is positive (or negative). This interesting result has not been mentioned by the previous work (see e.g. Gao et al., 1997, where the degree of freedom parallel to the crack axis was ignored). The validity of this prediction requires a carefully controlled experimental verification.

The electrical field is perpendicular to the crack axis

Next, consider an electrical field perpendicular to the crack axis. In this case, we have

$$K_{\mathrm{I}} = \sqrt{\pi a}\left[\sigma_{33}^{\infty} + \frac{C_T}{e}\frac{(c_{11}e_{33}-c_{13}e_{31})\sigma_{33}^{\infty}+(c_{33}e_{31}-c_{13}e_{33})\sigma_{11}^{\infty}}{c_{11}c_{33}-c_{13}^2} + \frac{C_T}{e}\frac{(c_{11}e_{33}^2+c_{33}e_{31}^2-2c_{13}e_{31}e_{33})E_3^{\infty}}{c_{11}c_{33}-c_{13}^2} + \frac{C_T}{e}\varepsilon_{33}E_3^{\infty}\right] \quad (46)$$

which includes an electric-field induced term proportional to the dielectric constant ε_{33}. Note that all coefficients of the applied electrical field E_3^{∞} on the right of (46) are positive; it is concluded that a positive electrical field, applied in the same direction as the poling direction, enhances the mode-I stress intensity factor, while a negative electrical field, applied opposite to the poling direction, reduces the mode-I stress intensity factor. Obviously, this result is in agreement with the experiments of Tobin and Pak (1993) and Park and Sun (1995). In connection with this, we note that some other experimental results have been reported by Kumar and Singh (1996), which show an opposite tendency. Hence, further experimental work is needed.

A pure electrical field perpendicular to the crack axis

This case is of particular interest. In the absence of mechanical load, the electric-field induced mode-I stress intensity factor is given by

$$K_{\mathrm{I}} = \sqrt{\pi a}\frac{C_T}{e}\left[\frac{(c_{11}e_{33}^2+c_{33}e_{31}^2-2c_{13}e_{31}e_{33})}{c_{11}c_{33}-c_{13}^2}+\varepsilon_{33}\right]E_3^{\infty} \quad (47)$$

Hence, an electrical field applied in the same direction as the poling produces a positive mode-I stress intensity factor. This gives an explanation for the observed electric-field induced crack growth in the direction perpendicular to the applied electrical field (see Cao and Evans, 1994; Lynch et al., 1995).

Another consequence of (47) is that an electrical field applied opposite to the poling direction produces a negative mode-I stress intensity factor. Despite the fact that it is in agreement with the spirit of the experimental results of Tobin and Pak (1993) and Park and Sun (1995), this sign effect has not been stated clearly in the previous work (see Gao et al. (1997), where the energy release rate criterion was used and the sign of the mode-I stress intensity factor was ignored). This result suggest that a crack perpendicular to the poling direction will never grow under an electrical field applied opposite to the poling direction. An experimental confirmation is needed for this prediction.

5. Conclusions

The effect of the saturation condition on the crack tip field and the stress intensity factors is examined in the paper for a general piezoelectric medium. It is found that the exact expression for the stress intensity factors can be obtained in some cases even without specifying the form of the saturation condition. In particular, it is the case when the mode-I stress intensity factor is examined for crack perpendicular to the poling axis in a poled ferroelectric. In this case, the results show that the electric-field induced part of the mode-I stress intensity factors is linearly dependent on the applied electrical field and that it changes sign when the direction of applied electrical field reverses. Thus, for a general poled ferroelectric, the strip-saturation model provides a clear and unified explanation for the aforementioned discrepancies between theory and experiments. We emphasize that these results are independent of the form of the saturation condition. For instance, a physically more reasonable saturation condition is: $|D| =$ constant along the strip S (because D_1 is non-uniform along S, "$D_2^+(x) = D_2^-(x) =$ constant" does not imply that $|D| =$ constant). All results obtained in the paper are valid for this strongly nonlinear saturation condition, because the condition $|D| =$ constant implies that $D_2^+(x)$ and $D_2^-(x)$ are bounded on the strip S.

Several interesting consequences of the strip-saturation model are pointed out in the paper. For instance, the results (45), (46) show that a tensile stress parallel to the crack axis makes a negative contribution to the mode-I stress intensity factor while a compressive stress makes a positive contribution. Furthermore, it is predicted that, based on the sign of the mode-I stress intensity factor, a crack perpendicular to the poling direction will never grow under an electrical field applied opposite to the poling direction. These results have not been stated in the previous related studies. The validity of these predictions needs further experimental verification.

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References

Cao, H. C. and Evans, A. G. (1994) Electric-field induced fatigue crack growth in piezoelectrics. *Journal of the American Ceramic Society* **77,** 1783–1786.

Dugdale, D. S. (1960) Yielding of steel sheets containing slits. *Journal of the Mechanics and Physics of Solids* **8,** 100–104.

Gao, H. J. and Barnett, D. M. (1996) An invariance property of local energy release rates in a strip saturation model of piezoelectric fracture. *International Journal of Fracture* **79,** R25–R29.

Gao, H. J., Zhang, T. Y. and Tong, P. (1997) Local and global energy release rates for an electrically yielded crack in a piezoelectric ceramic. *Journal of the Mechanics and Physics of Solids* **45,** 491–510.

Gong, X. and Suo, Z. (1996) Reliability of ceramic multilayer actuators: a nonlinear finite element simulation. *Journal of the Mechanics and Physics of Solids* **44,** 751–769.

Hao, T. H., Gong, X. and Suo, Z. (1996) Fracture mechanics for the design of ceramic multilayer actuators. *Journal of the Mechanics and Physics of Solids* **44,** 23–48.

Kumar, S. and Singh, R. N. (1996) Crack propagation in piezoelectric materials under combined mechanical and electrical loadings. *Acta Metallica* **44,** 173–200.

Lynch, C. S., Yang, W., Collier, L., Suo, Z. and McMeeking, R. M. (1995) Electric field induced cracking in ferroelectric ceramics. *Ferroelectrics* **166,** 11–30.

Mehta, K. and Virkar, A. V. (1990) Fracture mechanisms in ferroelectric–ferroelastic lead zirconate titanate (Zr: Ti = 0.54: 0.46) ceramics. *Journal of the American Ceramic Society* **73,** 567–574.

Muskhelishvili, I. N. (1963) *Some Basic Problems of the Mathematical Theory of Elasticity. P. Noordhoff Ltd., The Netherlands.*
Pak, Y. E. (1990) *Crack extension force in a piezoelectric material. Journal of Applied Mechanics* **57,** 647–653.
Pak, Y. E. (1992) Linear electro-elastic fracture mechanics of piezoelectric materials. *International Journal of Fracture* **54,** 79–100.
Pak, Y. E. and Tobin, A. (1993) On electrical field effects in fracture of piezoelectric materials, mechanics of electromagnetic materials and structures. AMD-Vol.161/MD-Vol. 42, ASME.
Park, S. and Sun, C. T. (1995) Fracture criteria for piezoelectric ceramics. *Journal of the American Ceramic Society* **78,** 1475–1480.
Pisarenko, G. G., Chushko, V. M. and Kovalev, S. P. (1985) Anisotropy of fracture toughness of piezoelectric ceramics. *Journal of the American Ceramic Society* **68,** 259–265.
Ru, C. Q. and Mao, X. (1997) Effect of microcracking on electric-field induced crack growth in dielectric ceramics. *Journal of the Mechanics and Physics of Solids* (To appear).submitted.
Ru, C. Q., Mao, X. and Epstein, M. (1997) Electric-field induced interfacial cracking in multilayer electrostrictive actuators. *Journal of the Mechanics and Physics of Solids* (To appear).
Sosa, H. (1992) On the fracture mechanics of piezoelectric solids. *International Journal of Solids and Structures* **29,** 2613–2622.
Stroh, A. N. (1958) Dislocations and cracks in anisotropic elasticity. *Philosophical Magazine* **3,** 625–646.
Sundar, V. and Newnham, R. E. (1992) Electrostriction and polarization. *Ferroelectrics* **135,** 431–446.
Suo, Z. (1993) Models for breakdown-resistent dielectric and ferroelectric ceramics. *Journal of the Mechanics and Physics of Solids* **41,** 1155–1176.
Suo, Z., Kuo, C.-M., Barnett, D. M. and Willis, J. R. (1992) Fracture mechanics for piezoelectric ceramics. *Journal of the Mechanics and Physics of Solids* **40,** 739–765.
Tobin, A. G. and Pak, Y. E. (1993) Effect of electric field on fracture behavior of PZT ceramics. *Proceedings of SPIE, Smart Structures and Materials*, **1916,** 78–86.
Yang, W. and Suo, Z. (1994) Cracking in ceramic actuators caused by electrostriction. *Journal of the Mechanics and Physics of Solids* **42,** 649–663.